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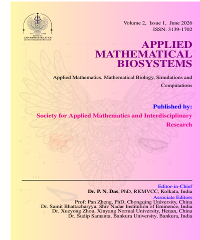

A Study on Central Semirings

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Abstract

The concept of the center plays a fundamental role in group theory, algebraic geometry, ring theory, and semiring theory. Extensive research has been devoted to the study of centers in semirings, resulting in significant contributions to the development of semiring theory. Motivated by the notion of the center of a semiring, we introduce the concept of a central semiring. The main objective of this paper is to investigate its structural properties of central semirings within the context of semiring theory and to derive several algebraic characterizations associated with them.

Keywords: *Semiring; Center of semiring; Mono semiring; Central semiring.*

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1. Introduction

In recent years, there has been a growing interest in the study of partially ordered and totally ordered algebraic structures, including semigroups, groups, semirings, semimodules, rings, and fields. A considerable portion of classical ring theory has been extended to the more general framework of semirings. Indeed, some mathematicians suggest that semirings represent a more fundamental class of algebraic structures, and that restricting attention solely to rings is analogous to limiting algebraic studies to the complex numbers.

The notion of a semiring was formally introduced in [1] by Vandiver in 1934, although semiring-like structures had already appeared implicitly in various areas of mathematics prior to that time. The systematic development of semiring theory began to gain momentum in the 1950s and has continued to evolve steadily since then. Furthermore, the theories of semigroups and rings have played a significant role in shaping the development of semirings and ordered semirings.

In the literature, a semiring we mean a non-empty set S together with two binary operations "$+$" and "$\cdot$" (usually denoted by juxtaposition) such that $(S,+)$ is a commutative semigroup and $(S,\cdot)$ is a semigroup, which are connected by ring-like distributivity. Thus, semirings provide a natural generalization of rings and distributive lattices. Semirings have significant applications across various branches of science and engineering [2]. Owing to this generality, semirings arise naturally in various applications, including automata theory, formal language theory, optimization theory, and other areas of applied mathematics. In the study of semirings and their representations, researchers frequently employ methods and techniques from ring theory and lattice theory, together with a variety of tools from categorical algebra and universal algebra. Useful resources on semiring theory are available in the references [3], [4], [5], [6], [7], [8].

Recent developments in semiring theory have focused on diverse directions, including ideal theory, closure operations, idempotent semirings, inverse semirings, tropical and topological semirings, and categorical approaches to semiring structures. For instance, the study of k-ideals, z-ideals and p-ideals has led to new insights into the algebraic and topological properties of semirings, while investigations of inverse semirings and functorial constructions have further expanded the structural framework of the subject. The theory of k-regular semirings has gained significant interest in recent years. S. Bourne [9] first introduced k-regular semirings as a generalization of regular rings to the semiring setting.

A semiring S is called a semiring with zero element '0' if $a+0=0+a=a$ and $0\cdot a=a\cdot 0=0$ for all $a\in S$. A semiring S is called a semiring with identity 1 if $1\cdot a=a\cdot 1=a$ for all $a\in S$.

A semiring may or may not have a zero and an identity element.

We consider a semiring $(S,+,\cdot)$ with zero element '0' and identity element '1' throughout this paper.

Let $(S,+,\cdot)$ and $(T,+,\cdot)$ be two semirings. Then a mapping $f : S \longrightarrow T$ is said to be a semiring homomorphism [8] of S into T if $f(x+y)=f(x)+f(y)$, $f(xy)=f(x)f(y)$ for all $x,y\in S$. An injective homomorphism is called a monomorphism, a surjective homomorphism is called an epimorphism and a bijective homomorphism is called an isomorphism.

The notion of center $Z(R)$ of a ring R is a substructure consisting of the elements x such that $xy=yx$ for all y in R. Moreover $Z(R)$ is a subring of the ring R but not necessarily an ideal of R. Several authors studied various center-like subsets for rings and under certain natural conditions. However, the study of centers within the framework of semirings has received relatively little attention so far.

Recent studies on center-like subsets of rings can be found in [10], [11], [12] . In 2006, M.K. Sen et al. defined [13] the term "Center of Semiring" to characterize a unique class of elements in a semiring. In [14], we introduced the notion of Birkhoff center of c-semiring and also in [15], we studied the structure of Birkhoff center of c-semiring. Moreover, in [16], an in-depth study of the structural properties of the center of a semiring was carried out. Motivated by these developments, we consider the class of semirings for which a semiring S coincides with its center $Z(S)$. To this end, we introduce a special class of semirings, called central semirings, in which $S = Z(S)$, and investigate several characterizations of this class. The main objective of this paper is to establish various results concerning central semirings that are analogous to those obtained in the corresponding context of ring theory.

2. Central Semiring

In the beginning of this section, we define the concept of the center of a semiring.

Definition 2.1. *[16] Let S be a semiring. A subset $Z(S)$ of a semiring S is called a center of S which is defined by $Z(S) = \{a \in S : ab = ba \text{ for all } b \in S\}$.*

With the help of the definition of center of semiring, we construct a special type of semiring, namely "Central of Semiring". Additionally, we present the definition of this semiring :

Definition 2.2. *A semiring S is said to be a central semiring if $Z(S) = S$.*

Thus, every element of a central semiring commutes multiplicatively with every other element.

To further clarify the concept, we present some examples of central semirings.

Example 2.3. *Let's consider the semiring $(\mathbb{N}, \oplus, \odot)$, where for any $b > a$, $a \oplus b = min\{a, b\} = a$ as addition on $\mathbb{N}$ and $a \odot b$ for the usual multiplication on $\mathbb{N}$. Then for any $b \in \mathbb{N}$, we have $ab = ba$, since multiplication on $\mathbb{N}$ is commutative. Consequently, we can conclude that $Z(\mathbb{N}) = \mathbb{N}$. Hence $\mathbb{N}$ is a central semiring.*

Example 2.4. *Consider the Boolean semiring $B = \{0, 1\}$; Define the operations "+" and "·" on B are given by $1 + 1 = 1$ and $1 \cdot 1 = 1$. Since multiplication in B is commutative, B forms a central semiring.*

Example 2.5. *Consider $S = \{0, 1, x\}$. Define the operations "+" and "." on S by means of the following tables:*

+	0	x	1
0	0	x	1
x	x	x	1
1	1	1	1

.	0	x	1
0	0	0	0
x	0	x	x
1	0	x	1

Then $(S, +, \cdot)$ is a semiring and $Z(S) = \{0, x, 1\} = S$. Therefore, S is a central semiring.

Example 2.6. *Consider $S = \{0, x, y, 1\}$. Define the operations "+" and "." on S by the following tables :*

+	0	x	y	1
0	0	x	y	1
x	x	x	y	1
y	y	y	y	1
1	1	1	1	1

.	0	x	y	1
0	0	0	0	0
x	0	x	x	x
y	0	x	y	y
1	0	x	y	1

Then $(S, +, \cdot)$ is a semiring and $Z(S) = \{0, x, y, 1\} = S$. Hence, $(S, +, \cdot)$ is a central semiring.

The following example illustrates a semiring which fails to be a central semiring.

Example 2.7. *We consider a semigroup $(M, \cdot)$ with multiplication table*

+	0	a	b
0	0	0	0
a	0	a	a
b	0	b	b

Let $S = Sub(M)$ be the set of all subsets of the semigroup M. Let us define '+' and '·' in S as : $A + B = A \cup B$ and $A \cdot B = \{ab \mid a \in A, b \in B\}$ for all $A, B \in S$. Then $(S, +, \cdot)$ is a semiring with zero element ϕ ; see [[5], Example 1.10]. Since M contains three elements, we obtain $|S| = 2^3 = 8$. Moreover, S is additively idempotent and multiplicatively idempotent noncommutative semiring. The center of S is given by $Z(S) = \{\{\phi\}, \{0\}\}$. Again we have $|Z(S)| = 2$. In this case, $Z(S) \neq S$. Consequently, S is not a central semiring.

3. Some Properties of Central Semiring

We examine in this section several foundational properties of central semirings, with particular emphasis on the relationship between commutative semirings and central semirings.

Proposition 3.1. *Every commutative semiring is a central semiring.*

Proof. Let S be a commutative semiring. By definition of commutative semiring, we have $ab = ba$ for all $a, b \in S$. We know that the center of semiring $Z(S) = \{a \in S : ab = ba \text{ for all } b \in S\}$. Let $a \in S$. Since S is commutative semiring, $ax = xa$ for every $x \in S$. This implies that $a \in Z(S)$. Since "a" is an arbitrary, it follows that every element of S is the element of $Z(S)$. Hence, $S \subseteq Z(S)$ (i).
For reverse part, we know that the center $Z(S)$ of a semiring S is a subsemiring of S, it follows that $Z(S) \subseteq S$ (ii).
From (i) and (ii), it follows that $Z(S) = S$. Thus, every element of S is central. Consequently, S is a central semiring. □

Definition 3.2. *[8]* A semiring S is called a mono semiring if $a + b = ab$ for all $a, b \in S$.

Theorem 3.3. *If S is a mono semiring, then S is a central semiring.*

Proof. Suppose that S is a mono semiring. We know that a semiring S is called central precisely when it coincides with its center $Z(S)$, that is, $S = Z(S)$. Thus, in order to conclude that S is a central semiring, we need to verify that $S = Z(S)$. By Theorem 2.5 of [16], the center $Z(S)$ of a semiring S is a subsemiring of S. This implies that $Z(S) \subseteq S$. It remains to prove that $S \subseteq Z(S)$. Let $a \in S$. Since S is a mono semiring then $a + b = ab$ for all $a, b \in S$. Now, for any $b \in S$, $ab = a + b = b + a = ba$. Thus, "a" commutes multiplicatively with every element of S, which implies that $a \in Z(S)$. Since "a" was arbitrary, we conclude that $S \subseteq Z(S)$. Hence $S = Z(S)$. Consequently, S is a central semiring. □

Definition 3.4. A variety of algebras is defined as a class of algebras of a fixed type that is closed under the operations of taking subalgebras, homomorphic images, and direct products (see [17]). According to Birkhoff's theorem, such a class constitutes a variety if and only if it can be characterized by a set of identities, i.e., it is an equational class. Consequently, the class of all semirings forms a variety. Moreover, the collection of all idempotent semirings also constitutes a variety, as shown in [18].

The subsequent goal is to verify that the collection of all central semirings is a variety. To achieve this, we start with the following lemma.

Lemma 3.5. *Let S be a central semiring and S' be an subsemiring of S. Then S' is a central semiring.*

Proof. Let S be a central semiring, so that $Z(S) = S$. We aim to prove that any subsemiring S' is a central semiring, denoted as $Z(S') = S'$. Suppose $a \in S' \subseteq S = Z(S)$. Consequently, $ab = ba$ for all $b \in S$. As S' is a subsemiring of S, we can infer that $ab = ba$ for all $b \in S'$. Hence, $a \in Z(S')$, indicating that $S' \subseteq Z(S')$. On the other hand, since $Z(S')$ serves as a subsemiring of S', it follows that $Z(S') \subseteq S'$. Thus, we conclude that $Z(S') = S'$, affirming that S' is a central semiring. □

Lemma 3.6. *Every homomorphic image of a central semiring is also a central semiring*

Proof. Let S be an almost idempotent central semiring with identity 1_S and S' be a central semiring with identity $1_{S'}$. Let $f : S \to S'$ be onto homomorphism. Then S' is the homomorphic image of the central semiring S. We have to show that S' is an almost idempotent semiring. Since f is an onto homomorphism, so $S' = \{f(a) : a \in S\}$. Furthermore, $f(1_S) = 1_{S'}$ acts as the identity element of S'. Let $s' \in S'$. Then there exists $a \in S$ such that $f(a) = s'$. As S is a semiring, we know that $ab = ba$ for all $a, b \in S$. So, for any $f(b) \in S'$, we have $s'f(b) = f(a)f(b) = f(ab) = f(ba) = f(b)f(a) = f(b)s'$. Therefore, we conclude that $s' = f(a) \in Z(S') = S'$. Since s' is an arbitrary element of S', it follows that S' is a central semiring. □

Lemma 3.7. *Let $\{S_i : i = 1, 2, ..., n\}$ be a finite family of semirings. Then the direct product of semirings $S = \prod_{i=1}^{n} S_i$ is central semiring if and only if each semiring S_i is central semiring.*

Proof. Let's consider a family of semirings $\{S_i : i = 1, 2, ..., n\}$, where each semiring S_i is central semiring. Now, suppose we have an element $(x_1, x_2, ..., x_n) \in S$, where each $x_i \in S_i$. Since each S_i is central semiring, for any $y_i \in S_i$, we have $x_i y_i = y_i x_i$ for all $i = 1, 2, ..., n$. Consequently, it follows that $(x_1, x_2, ..., x_n)(y_1, y_2, ..., y_n) = (x_1y_1, x_2y_2, ..., x_ny_n)$ $= (y_1x_1, y_2x_2, ..., y_nx_n) = (y_1, y_2, ..., y_n)(x_1, x_2, ..., x_n)$. As a result, we conclude that S is a semiring.

Conversely, suppose that $S = \prod_{i=1}^{n} S_i$ is central semiring. Our objective is to demonstrate that each semiring S_i is central semiring. To accomplish this, we will examine the mapping $\pi : S \to S_i$ defined by $\pi((x_1, x_2, ..., x_n)) = x_i$ for all $(x_1, x_2, ..., x_n) \in S$. It can be observed that π is an onto homomorphism from $S = \prod_{i=1}^{n} S_i$ to S_i. Consequently, according to Lemma 3.6, S_i is a idempotent central semiring. for all $i = 1, 2, ..., n$. This completes the proof. □

We now present an example illustrating the above lemma.

Example 3.8. *Consider the semiring $S = \mathbb{N}_0 \times \mathbb{N}_0$. Since $\mathbb{N}_0$ is a central semiring, it follows from Lemma 3.7 that S is also a central semiring.*

Theorem 3.9. *The class of all central semirings is a variety.*

Proof. By utilizing the Lemmas 3.5, 3.6 and 3.7, we have proved that the class of almost idempotent central semirings is closed under taking subsemirings, homomorphic images and direct products. Therefore, it can be concluded that the class of all central semirings is a variety. □

4. Application of Central Semiring

The concept of a central semiring plays a significant role in the structural analysis of semirings and their algebraic properties. A central semiring, characterized by the presence of elements that commute multiplicatively with all other elements, provides a natural generalization of the notion of the center in classical ring theory. This structure facilitates the study of commutativity conditions and symmetry within non-ring algebraic systems.

One of the primary applications of central semirings arises in the simplification of algebraic computations. The existence of central elements enables the decomposition of complex expressions into more manageable forms, thereby aiding in the analysis of semiring homomorphisms and congruences. In particular, central elements often serve as fixed points under various mappings, which proves useful in the classification of semiring structures.

Central semirings also find applications in automata theory and theoretical computer science, where semirings are widely used to model weighted automata, formal languages, and optimization problems. The presence of a non-trivial center allows for more efficient computation in path problems, such as shortest path algorithms, by reducing non-commutative complications in weight aggregation.

Semiring structures frequently arise in fuzzy set theory and fuzzy logic. In fuzzy systems, algebraic operations are often modeled through idempotent semirings. Central semirings are particularly useful in fuzzy algebra because commutativity ensures consistency in aggregation processes and fuzzy relational compositions

In addition, central semirings contribute to the study of matrix semirings and linear algebra over semirings. When central elements are involved, certain matrix operations exhibit properties analogous to those in commutative settings, thereby extending classical results to broader contexts. This has implications in areas such as tropical algebra, fuzzy systems, and decision-making models.

Furthermore, central semirings are instrumental in the investigation of ideals and radicals in semiring theory. The interaction between central elements and ideals often leads to refined characterizations of prime and maximal ideals, enhancing the understanding of the internal structure of semirings. These insights are valuable in both pure and applied mathematical research.

5. Conclusion

In this paper, we established that every commutative semiring is a central semiring and that every mono semiring is likewise central. We further proved that the class of all central semirings is closed under subsemirings, homomorphic images, and finite direct products, and consequently forms a variety. The results presented here provide a foundational framework for the study of central semirings and suggest several directions for future research. In particular, it would be worthwhile to characterize central semirings within significant subclasses of semirings, investigate their ideal-theoretic and structural properties, and explore centrality in matrix semirings and polynomial semirings. Moreover, the present work encourages further examination of the relationships between central semirings and other center-related notions that arise in semiring theory.

In conclusion, the study of central semirings not only deepens the theoretical foundations of semiring theory but also supports a wide range of applications across mathematics and computer science. Their ability to bridge commutative and non-commutative frameworks makes them a powerful tool in modern algebraic investigations. Future research may focus on the classification of central semirings under additional constraints, the exploration of their categorical properties, and their potential applications in emerging areas of mathematics and computer science.

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